MID-SEMESTER EXAM — ISI BANGALORE — 17TH SEPT 2012

INTRODUCTION TO DIFFERENTIAL GEOMETRY — 100 POINTS

Notes.
(a) Begin each answer on a separate sheet.
(b) Justify all your steps. Assume only those theorems that have been proved in class. All other steps should be justified.
(c) $\mathbb{R}$ = real numbers. For a curve we use κ = curvature, τ = torsion.

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1. [24 points] Let $F(x, y, z) = (3x^2y + e^z - 1, 2x + z\cos y)$ so that F defines a C^∞ map $\mathbb{R}^3 \to \mathbb{R}^2$. Let u, v denote coordinates in $\mathbb{R}^2$.
   (i) Find a paramerized curve $\vec{\alpha}\colon \mathbb{R} \to \mathbb{R}^3$ such that the curve $F \circ \vec{\alpha}$ passes through the origin $(0, 0)$ with velocity $(1, 1)$.
   (ii) Prove that there exists a parametrized curve $\vec{\beta}\colon I \to \mathbb{R}^3$ such that the intersection of $(F \circ \vec{\beta})(I)$ with the v-axis contains an open interval in the v-axis.
   (iii) Prove that $F^{-1}((0, 0))$ is a regular parametrizable curve near the origin $(0, 0, 0)$.

2. [16 points]
   (i) Let $\vec{\gamma}(t)$ be a regular curve in $\mathbb{R}^n$. Prove that the curvature $\kappa(t)$ is given by the square-root of the following determinant:

$$\begin{vmatrix} \dot{\vec{\gamma}} \cdot \dot{\vec{\gamma}} & \dot{\vec{\gamma}} \cdot \ddot{\vec{\gamma}} \\ \dot{\vec{\gamma}} \cdot \ddot{\vec{\gamma}} & \ddot{\vec{\gamma}} \cdot \ddot{\vec{\gamma}} \end{vmatrix}$$

   (Hint: First verify that the formula doesn't change under reparametrization and then evaluate it using a unit-speed parametrization.)
   (ii) Let $f(t)$ be a C^∞ function on $\mathbb{R}$. Find the curvature of the curve $\vec{\gamma}(t) = (t, f(t))$ at $t = 0$.

3. [20 points] Let $\vec{\gamma}(s)$ be a unit-speed curve in $\mathbb{R}^3$ with $\kappa(s) > 0$ and $\tau(s) \neq 0$ for all s.
   (i) If γ lies on the surface of a sphere in $\mathbb{R}^3$, prove that

$$\frac{\tau}{\kappa} = \frac{d}{ds}\left(\frac{\dot{\kappa}}{\tau\kappa^2}\right).$$

   (Hint: Differentiate repeatedly the relation $(\vec{\gamma} - \vec{a}) \cdot (\vec{\gamma} - \vec{a}) = R^2$ where R is the radius and $\vec{a}$ the centre of the sphere.)

(ii) Conversely, if the above relation holds, prove that $\rho^2 + (\dot{\rho}\sigma)^2$ is a positive constant where $\rho(s) = 1/\kappa(s)$ and $\sigma(s) = 1/\tau(s)$. Calling this constant R, deduce that $\vec{\gamma}(s)$ lies on a sphere of radius R.

(Hint: Consider $\vec{\gamma} + \rho\vec{n} + \dot{\rho}\sigma\vec{b}$ where $\vec{n}(s)$ and $\vec{b}(s)$ denote the principal normal and binormal respectively.)

4. [15 points] Give an example of a regular parametrized curve in $\mathbb{R}^3$ that has constant curvature 1 and constant torsion -1 everywhere. Write down formulas for the unit tangent, principal normal and binormal vectors for each point on the curve.

5. [10 points] Let $p, q \in C^\infty(\mathbb{R})$. Find a diffeomorphism $F(x, y)\colon \mathbb{R}^2 \to \mathbb{R}^2$, that takes the locus of $y = p(x)$ to the locus of $x = q(y)$.

6. [15 points] Let $f(x, y, z) \in C^\infty(U)$ where U is a neighborhood of $\vec{0} \in \mathbb{R}^3$. Assume $f(\vec{0}) = 0$ and $f_x(\vec{0}) \neq 0$. Prove that there exists a neighborhood $U' \subset U$ of $\vec{0}$ where the locus of $f(x, y, z) = 0$ is the graph of a function $h(y, z) \in C^\infty(V)$ for some neighborhood V of the origin in $\mathbb{R}^2$. Thus

$$\{(x, y, z) \mid f(x, y, z) = 0\} \cap U' \quad = \quad \{(h(y, z), y, z) \mid (y, z) \in V\}.$$